

Exact null controllability, complete stabilizability and exact final observability: the case of neutral type systems

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Abstract

For abstract linear systems in Hilbert spaces we revisit the problems of exact controllability and complete stabilizability (stabilizability with an arbitrary decay rate), the latter property is equivalent to exact null controllability. We extend this result to the case when the feedback is not bounded. This enables the characterization of exact null controllability and complete stabilizability for neutral type systems. By duality, we obtain a result about continuous final observability. Illustrative examples are given.

1 Introduction

Consider the controlled neutral type system

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + Lz_t(\cdot) + Bu(t), \quad (1)$$

where

$$Lz_t(\cdot) = \int_{-1}^0 [A_2(\theta)\dot{z}(t+\theta) + A_3(\theta)z(t+\theta)] d\theta,$$

with $z(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and the matrices A_{-1} , A_2 , A_3 and B are of appropriate dimension. The elements of A_2 and A_3 take values in $L_2(-1, 0)$.

System (1) may be represented in a Hilbert space by the equation

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \quad (2)$$

where $\mathcal{B}u = (Bu, 0)$ and \mathcal{A} is the infinitesimal generator of a C_0 -semigroup $e^{\mathcal{A}t}$ given in the product space $M_2 \stackrel{\text{def}}{=} \mathbb{R}^n \times L_2(-1, 0; \mathbb{R}^n)$ and defined by

$$\mathcal{A}x(t) = \begin{pmatrix} Lz_t(\cdot) \\ \frac{dz_t(\theta)}{d\theta} \end{pmatrix}, \quad x(t) = \begin{pmatrix} v(t) \\ z_t(\cdot) \end{pmatrix},$$

with the domain $D(\mathcal{A}) = \{(v, \varphi(\cdot)) : \varphi(\cdot) \in H^1, v = \varphi(0) - A_{-1}\varphi(-1)\}$.

Our purpose is to characterize the exact null controllability of delay systems of neutral type given by equation (1), to show that this notion is equivalent to the complete stabilizability (exponential stabilizability with an arbitrary decay rate) of the system and, by duality, to give

conditions of the exact final observability of such a system with an output $y(t) = Cz(t)$ or $y(t) = Cz(t-1)$, where $y(t)$ take values in \mathbb{R}^p .

The problem of exact controllability for systems of neutral type has been widely investigated. References and important results for system (1) can be found in [24]. A simplification and precision of the proofs are given in [20]. The duality with exact (continuous) observability is analyzed in [18]. For the stabilizability problem, after the first important works [15, 14], there were many results on the stabilizability of delay systems (see, for example, [27, 13] and references therein) but neutral type systems have been less investigated [30, 17]. In [7] the main scheme of stabilizing neutral type systems and the robustness, with respect to the delays, of the stabilizing feedback were analyzed. The problem of asymptotic non-exponential stabilizability, which appears only for neutral type systems, was treated in [22, 25].

This paper is organized as follows. In Section 2 we give the results on the equivalence between exact null controllability and complete stabilizability for abstract systems in Hilbert spaces; first for the case of bounded input and feedback and then for admissible input and feedback. In Section 3, we give the main result on the equivalence between exact null controllability and complete stabilizability for neutral type systems, as well as we give a characterization of these concepts. Section 4 is concerned with the dual notion of observability: final continuous observability.

2 Preliminary results

In this section we consider the abstract system

$$\dot{x} = \mathcal{A}x + \mathcal{B}u \quad (3)$$

where \mathcal{A} , with domain $D(\mathcal{A})$ is the generator of a C_0 -semigroup $e^{\mathcal{A}t}$ in the Hilbert space X and \mathcal{B} is a linear operator, which may be unbounded but admissible (see, for example, [32]), from the Hilbert space U to X .

2.1 Bounded input and feedback

Let us first suppose that \mathcal{B} is bounded. The solution of the system (3) with the initial condition x_0 and the control $u(t) \in L_2^{\text{loc}}(\mathbb{R}^+; U)$ is given by

$$x(t) = e^{\mathcal{A}t}x_0 + \int_0^t e^{\mathcal{A}(t-\tau)}\mathcal{B}u(\tau)d\tau.$$

The following definition are well known.

Definition 2.1 *System (3) is said to be exactly controllable at time T if for all $x_0, x_1 \in X$, there is a control $u(t) \in L_2(0, T; U)$ such that the corresponding solution of the system verifies $x(T) = x_1$. The system is said to be exactly null controllable if in the preceding definition $x_1 = 0$.*

There are several results about exact (null) controllability. For example, it is well known that if \mathcal{B} is compact, particularly if U is finite dimensional, then there is no exact controllability. Another condition of exact controllability, in the case of the bounded operator \mathcal{B} , is that for all $t \geq 0$, the operator $e^{\mathcal{A}t}$ is onto (surjective) [10].

In what follows, we need the following criteria of exact (null) controllability [2].

Theorem 2.2 *System (3) is exactly null controllable at time T if and only if*

$$\exists \delta > 0 : \quad \forall x \in X, \quad \int_0^T \|B^* e^{A^* t} x\|^2 dt \geq \delta^2 \|e^{A^* T} x\|^2.$$

For the condition of exact controllability, the operator $e^{A^ T}$ must be replaced by the identity I in the right part of the inequality.*

The characterization of exact null controllability is due to of a result on range inclusion in Hilbert spaces [3].

We also need some notions of stabilizability.

Definition 2.3 *System (3) is said to be exponentially stabilizable if there is a linear bounded operator feedback \mathcal{F} such that the semigroup $e^{(A+B\mathcal{F})t}$ is exponentially stable: there is a $\omega > 0$ such that*

$$\|e^{(A+B\mathcal{F})t}\| \leq M_\omega e^{-\omega t}, \quad M_\omega \geq 1. \quad (4)$$

The system is said to be completely stabilizable (or stabilizable with an arbitrary decay rate) if for all $\omega \in \mathbb{R}$ there is a linear bounded feedback \mathcal{F} such that (4) holds.

The relation between exact controllability and stabilizability is as follows : exact null controllability implies exponential stabilizability. If e^{At} is a group, complete stabilizability implies exact controllability (see, for example, [34]). The latter result by Zabczyk was extended in [23, 35] for the case of a semigroup e^{At} provided that the operators e^{At} are surjective for all $t \geq 0$.

We can now extend this latter result to exact null controllability under some additional condition. In order to explain this additional condition, let us give an example of a semigroup with rapid exponential stability and without control.

Example 2.4 [26, 31] *In the space $L_2(0, +\infty)$, consider the semigroup is*

$$S(t)f(x) = e^{-\frac{t^2}{2} - xt} f(x+t), \quad t \geq 0, \quad x \geq 0.$$

It is not difficult to see that for this semigroup, for all $\omega > 0$, there is a constant $M_\omega \geq 1$ such that $\|S(t)\| \leq M_\omega e^{-\omega t}$. We have also $\sigma(T(t)) = \{0\}$ and then the spectrum of the infinitesimal generator is empty. In the other hand, there is initial conditions f such that $S(t)f \neq 0$ for any $t \geq 0$.

To avoid this situation, we introduce the following assumption.

Assumption 2.5 (Assumption A) *Let us suppose that the system is completely stabilizable. We say that the system verifies the assumption A if there is a $t_0 > 0$ such that $M_\omega e^{-\omega t_0} \rightarrow 0$ as $\omega \rightarrow \infty$.*

This is possible, for example, if M_ω is bounded by a constant or if it is polynomial in ω . For the Example 2.4, one can show that $M_\omega \geq e^{\omega + \frac{1}{2}}$. Assumption A is also verified if $S(t)$ is a group, or a surjective semigroup [34, 5].

Theorem 2.6 *System (3) with bounded a operator \mathcal{B} is completely stabilizable by bounded feedbacks \mathcal{F} and verifies Assumption A, then it is exactly null controllable. If the system is exactly null controllable, then it is completely stabilizable*

Proof. Suppose that the system is exactly null controllable at time T . Then

$$\forall x_0 \in X, \quad \exists u(\cdot) \in L_2(0, T; U) : \quad x(T, x_0, u(\cdot)) = 0,$$

where $x(t) = x(t, x_0, u(\cdot))$ is the solution with the initial condition x_0 and the control $u(t)$:

$$x(t, x_0, u(\cdot)) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}\mathcal{B}u(\tau)d\tau.$$

Then

$$\forall x_0 \in X, \quad \exists u(\cdot) \in L_2(0, \infty; U) : \int_0^{+\infty} (\|x(t)\|^2 + \|u(t)\|^2) dt < \infty.$$

This means that the system is exponentially stabilizable [34, Th. 3.3] :

$$\exists F \in \mathcal{L}(U, X) : \quad \left\| e^{(A+\mathcal{B}F_0)t} \right\| \leq M_{\omega_0} e^{-\omega_0 t}, \quad \omega_0 > 0.$$

On the other hand, the exact null controllability of system (3) is equivalent to the exact null controllability of the system

$$\dot{x} = (\mathcal{A} + \omega I)x + \mathcal{B}u, \quad \omega \in \mathbb{R}.$$

This means that for all $\omega > 0$, we have for some $\mu_\omega > 0$,

$$\exists \mathcal{F}_\omega \in \mathcal{L}(U, X) : \quad \left\| e^{(\mathcal{A}+\mathcal{B}\mathcal{F}_\omega)t} \right\| \leq M_{\mu_\omega} e^{-(\mu_\omega+\omega)t} \leq M_\omega e^{-\omega t}.$$

Suppose now that the system is completely stabilizable but it is not exactly null controllable for any $T > 0$. Let us fix $T > 0$, sufficiently large to verify Assumption A. From Theorem 2.2, the lack of exact null controllability implies the existence of a normed sequence $\{x_n, n \in \mathbb{N} : \|x_n\| = 1\}$ such that

$$\lim_{n \rightarrow \infty} \int_0^T \left\| \mathcal{B}^* e^{\mathcal{A}^* t} x_n \right\|^2 dt = 0,$$

when $e^{\mathcal{A}^* T} x_n \not\rightarrow 0$. We can conclude that there is ε_0 such that $\|e^{\mathcal{A}^* T} x_n\| > \varepsilon_0$ (in fact a subsequence of $\{x_n\}$, but we can suppose that it is the required sequence).

If \mathcal{F} is a bounded feedback then we have (see, for example [2]), for all $x \in X$:

$$e^{(\mathcal{A}+\mathcal{B}\mathcal{F})t}x = e^{\mathcal{A}t}x + \int_0^t e^{\mathcal{A}(t-\tau)}\mathcal{B}\mathcal{F}e^{(\mathcal{A}+\mathcal{B}\mathcal{F})\tau}x d\tau, \quad t \geq 0, \quad (5)$$

and then

$$e^{(\mathcal{A}^*+\mathcal{F}^*\mathcal{B}^*)t}x = e^{\mathcal{A}^*t}x + \int_0^t e^{(\mathcal{A}^*+\mathcal{F}^*\mathcal{B}^*)\tau}\mathcal{F}^*\mathcal{B}^*e^{\mathcal{A}^*(t-\tau)}x d\tau, \quad t \geq 0. \quad (6)$$

Let us now find $\omega_0 > 0$ and \mathcal{F}_0 such that

$$\left\| e^{(\mathcal{A}^*+\mathcal{F}_0^*\mathcal{B}^*)T} \right\| \leq M_{\omega_0} e^{-\omega_0 T} \leq \frac{\varepsilon_0}{2}.$$

This is possible because we assumed that the system verifies Assumption A, T must be sufficiently large to verify this assumption. As T and \mathcal{F}_0 are now fixed, we have

$$\left(\int_0^T \left\| e^{(\mathcal{A}^*+\mathcal{F}_0^*\mathcal{B}^*)\tau}\mathcal{F}^*\mathcal{B}^* \right\|^2 d\tau \right)^{\frac{1}{2}} \leq C_0,$$

where $C_0 > 0$ is a constant.

Now let N be an integer such that for $n > N$ we have

$$\int_0^T \|B^* e^{A^* t} x_n\|^2 dt < \frac{\varepsilon_0}{2C_0}.$$

This gives

$$M_{\omega_0} e^{-\omega_0 T} \geq \|e^{(A^* + \mathcal{F}^* \mathcal{B}^*)t} x_n\| > \varepsilon_0 - C_0 \frac{\varepsilon_0}{2C_0} = \frac{\varepsilon_0}{2},$$

which is in contradiction with the inequality $M_{\omega_0} e^{-\omega_0 T} \leq \frac{\varepsilon_0}{2}$.

This means that there is some T for which null exact controllability holds. ■

2.2 Unbounded input and feedback operators

For several classes of control systems, the input operator \mathcal{B} may not be bounded and it is very restrictive to assume that the feedback operator \mathcal{F} is bounded. For a general theory on systems with unbounded control and observation we refer to the paper [29]. For the subclass of interest, which includes linear neutral type systems and for our purpose we refer to [17] and [1, 5]).

As our final goal is to analyze exact null controllability and complete stabilizability for neutral type systems, we will now consider a wider context of systems with unbounded input and output operators.

Let $X_1 = D(\mathcal{A})$ with the graph norm noted $\|x\|_1$ and X_{-1} the completion of the space X with respect to the resolvent norm $\|x\|_{-1} = \|(\lambda I - \mathcal{A})^{-1}x\|_X$. We have the following relation

$$X_1 \subset X \subset X_{-1}, \quad (7)$$

with continuous dense injections.

Definition 2.7 Let \mathcal{B} be a linear operator, bounded from the Hilbert space U to X . We say that \mathcal{B} is an admissible input operator for the semigroup e^{At} if there exists t_1 such that

$$\int_0^{t_1} e^{A(t_1-\tau)} \mathcal{B}u(\tau) d\tau \in X_1,$$

and for some $\beta > 0$

$$\left\| \int_0^{t_1} e^{A(t_1-\tau)} \mathcal{B}u(\tau) d\tau \right\|_{X_1} \leq \beta \|u\|_{L_2(0,t_1)}.$$

Definition 2.8 Assume that operator \mathcal{F} is a linear operator, bounded from X_1 to the Hilbert space Y . We say that it is an admissible output operator for the semigroup e^{At} if there exists t_1 such that for some $\alpha > 0$

$$\left\| \mathcal{F} e^{A(t_1-\tau)} x \right\|_{L_2(0,t_1)} \leq \alpha \|x\|_X, \quad x \in X_1.$$

Admissibility for some t_1 implies admissibility for all $t_1 > 0$ (see for example [1, 5]). From the general result on the perturbation of the semigroup for the Pritchard-Salamon class we can deduce the following Cauchy formula for the perturbed semigroup $e^{(A+\mathcal{B}\mathcal{F})t}$ for admissible input and output operators \mathcal{B} and \mathcal{F} respectively:

$$e^{(A+\mathcal{B}\mathcal{F})t} x = e^{At} x + \int_0^t e^{A(t-\tau)} \mathcal{B} \mathcal{F} e^{(A+\mathcal{B}\mathcal{F})\tau} x d\tau, \quad (8)$$

for all $x \in X_1$. Moreover $e^{(A+\mathcal{B}\mathcal{F})t}$ extends to a C_0 -semigroup on X .

This means that Theorem 2.6 may be reformulated for an admissible input operator and admissible output feedback.

Theorem 2.9 *If the system (3) with an admissible operator \mathcal{B} is completely stabilizable by an admissible A -bounded feedbacks \mathcal{F} and verifies Assumption A, then it is exactly null controllable. If the system is exactly null controllable, then it is completely stabilizable.*

Proof. In [1, Theorem 5.5] (see also [5]), in a more general situation, it is shown that system (3) with an admissible operator \mathcal{B} is exponentially admissibly stabilizable (in X_1 and X) if and only if it is exponentially stabilizable by a bounded feedback. Hence, we can suppose without loss of generality, that in (8) the operator \mathcal{F} is bounded: $\mathcal{F} \in \mathcal{L}(X, U)$. This means that complete stabilizability by admissible feedbacks holds if and only if there is complete stabilizability by bounded feedbacks. Then we can use the proof of Theorem 2.6 to conclude. ■

2.3 A technical Lemma

In the next section we need the following lemma.

Lemma 2.10 *Let A be a $(n \times n)$ -matrix and B a $(n \times m)$ -matrix. The following statements are equivalent.*

1. $\text{rank}(\lambda I - A \quad B) = n$, for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$.
2. $\text{rank}(B \quad AB \quad \dots \quad A^{n-1}B) = \text{rank}(B \quad AB \quad \dots \quad A^{n-1}B \quad A^n)$, which is equivalent to the inclusion:

$$\text{Im } A^n \subset \text{Im}(B \quad AB \quad \dots \quad A^{n-1}B).$$

Proof. Conditions 1 and 2 of the lemma may be formulated as follows.

- 1b. If there is $x \neq 0$ such that $A^*x = \lambda x$ and $B^*x = 0$, then $\lambda = 0$.
- 2b. If $x \neq 0$ is such that $B^*A^{*i}x = 0$, $i \in \mathbb{N}$, then $A^{*n}x = 0$.

Suppose that 1b holds. Let $x \neq 0$ be such that $B^*A^{*i}x = 0$ for all integers i . Then the subspace generated by x is A^* invariant and contained in $\text{Ker } B^*$. It contains an eigenvector of A^* . This means that x is an eigenvector of A^* : $A^*x = \lambda x$. The condition 1b implies that $\lambda = 0$. Then $A^{*n}x = \lambda^n x = 0$. This gives 2b.

Suppose now that 2b holds. Let $x \neq 0$ be such that $A^*x = \lambda x$ and $B^*x = 0$. This implies that $B^*A^{*i}x = \lambda^i B^*x = 0$, for all $i \in \mathbb{N}$. From condition 2b, we obtain that

$$0 = A^{*n}x = \lambda^n x.$$

As $x \neq 0$, this implies that $\lambda = 0$. This gives statement 1b. ■

3 The neutral type systems: controllability and stabilizability

Our purpose is to show, for the neutral type system given by (1), that exact null controllability is equivalent to complete stabilizability by adequate feedback. We make use of the results of the preceding section and also provide a characterization of these properties. The relation between

exact controllability and exponential stabilizability for linear neutral type systems may be found in several papers (see, for example [30, 8, 14, 4] and references therein).

For the analysis of stabilizability, we need the structure of the spectrum of the state operator \mathcal{A} of system (1) and the condition of the growth of semigroup e^{At} .

Theorem 3.1 [21] *Let $\Delta_{\mathcal{A}}$ be the matrix:*

$$\Delta_{\mathcal{A}}(\lambda) = \lambda I - \lambda e^{-\lambda} A_{-1} - \lambda \int_{-1}^0 e^{\lambda s} A_2(s) ds - \int_{-1}^0 e^{\lambda s} A_3(s) ds. \quad (9)$$

The spectrum of \mathcal{A} , noted $\sigma(\mathcal{A})$ consists of eigenvalues only which are the roots of the equation $\det \Delta_{\mathcal{A}}(\lambda) = 0$. The corresponding eigenvectors of \mathcal{A} are of the form

$$\begin{pmatrix} v - e^{-\lambda} A_{-1} v \\ e^{\lambda \theta} v \end{pmatrix}, \quad v \in \text{Ker } \Delta_{\mathcal{A}}(\lambda).$$

The spectrum is countable and the semigroup e^{At} verifies the spectrum growth assumption (see, for example, [6]):

$$\forall \omega > \omega_0 = \sup \text{Re } \sigma(\mathcal{A}), \quad \exists M_{\omega}, \quad \|e^{At}\| \leq M_{\omega} e^{\omega t}.$$

Definition 3.2 *System (1) is exactly null controllable if for some $T > 0$ and for all $x_0 \in D(\mathcal{A})$, there is a control $u(\cdot) \in L_2(0, T; \mathbb{R}^m)$ such that*

$$e^{At} x_0 + \int_0^T e^{A(T-\tau)} \mathcal{B} u(\tau) x_0 d\tau = 0,$$

this correspond to the concept of complete controllability given first by N. N. Krasovskiĭ for retarded systems.

Let \mathcal{R}_T be the linear operator defined by

$$\mathcal{R}_T u(\cdot) = \int_0^T e^{A(T-\tau)} \mathcal{B} u(\tau) x d\tau, \quad u(\cdot) \in L_2(0, T; \mathbb{R}^m).$$

The operator \mathcal{R}_T is bounded: $\mathcal{R}_T \in \mathcal{L}(L_2(0, T; \mathbb{R}^m), X)$. Moreover it takes value in $D(\mathcal{A})$ and is bounded from $L_2(0, T; \mathbb{R}^m)$ to X_1 (see [8, Corollary 2.7] and [24] for our system). The null exact controllability may be formulated as

$$e^{AT}(X_1) \subset \text{Im } \mathcal{R}_T,$$

where $e^{AT}(X_1)$ is the image of $D(\mathcal{A})$ by the operator e^{AT} and $\text{Im } \mathcal{R}_T$ is the image of $L_2(0, T; \mathbb{R}^m)$ by \mathcal{R}_T . From the well-known characterization of range inclusion in Hilbert spaces [3] we can obtain the following proposition, which is an extension of Theorem 2.2.

Proposition 3.3 *System (1) is exactly null controllable for some $T > 0$ if and only if there is a constant $\delta > 0$ such that*

$$\int_0^T \left\| \mathcal{B}^* e^{A^*(T-\tau)} x \right\|_{\mathbb{R}^m}^2 d\tau \geq \delta^2 \left\| e^{A^* T} x \right\|_{X_1^*}^2, \quad (10)$$

for all $x \in X_1^$. Here X_1^* is the space X_{-1}^d , which is the completion of the space $X = M_2$ with respect to the norm $\|(\lambda I - \mathcal{A}^*)^{-1} x\|$.*

We can now give the main result of this Section.

Theorem 3.4 *The following statements are equivalent:*

1. System (1) is exactly null controllable,
2. The following two conditions hold
 - 2.1. $\text{rank}(\Delta_A(\lambda) \ B) = n$ for all $\lambda \in \mathbb{C}$,
 - 2.2. $\text{rank}(\mu I - A_{-1} \ B) = n$ for all $\mu \in \mathbb{C}$, $\mu \neq 0$,
3. System (1) is completely stabilizable by a feedback law of the form

$$u(t) = F_{-1}\dot{z}(t-1) + Fz_t(\cdot), \quad (11)$$

where $Fz_t(\cdot) = \int_{-1}^0 [F_2(\theta)\dot{z}(t+\theta) + F_3(\theta)z(t+\theta)] d\theta$.

Proof. For the proof of the theorem we follow the scheme $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

STEP 1. Suppose that system (1) is exactly null controllable. The necessity of condition 2.1 is trivial. Let us show that the 2.2 is verified. We follow a method used in [12].

Then, for some T , for all initial conditions $\varphi \in H^1(-1, 0; \mathbb{R}^n)$, there is a control $u(\cdot) \in L_2(0, T; \mathbb{R})$, $u(t) = 0$ for $t > T$, such that $z(t) = 0$, $t > T$. We can suppose that $T > n$.

We have

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + Lz_t + Bu(t).$$

Replacing $\dot{z}(t-1)$ in this equation, we obtain

$$\dot{z}(t) = A_{-1}(A_{-1}\dot{z}(t-2) + Lz_{t-1} + Bu(t-1)) + Bu(t).$$

Repeating this procedure, we obtain

$$\dot{z}(t) = A_{-1}^N \dot{z}(t-N) + \sum_{k=0}^{N-1} A_{-1}^k (Lz_{t-k} + Bu(t-k)). \quad (12)$$

Putting $t = N \geq T$, and using the continuity of $z(t)$ we obtain

$$A_{-1}^N (\dot{z}(+0) - \dot{z}(-0)) + \sum_{k=0}^{N-1} A_{-1}^k (Bu(N-k+0) - Bu(N-k-0)). \quad (13)$$

As $z(t)$ for $t > 0$ is the solution of equation (1) we have

$$\dot{z}(+0) = A_{-1}z(-1) + Lz(+0) + Bu(+0),$$

then, replacing this expression in (13), and putting $z_0(\theta) = \varphi(\theta)$, we obtain

$$A_{-1}^N (A_{-1}\varphi(-1) + L\varphi(\theta) - \dot{\varphi}(-0)) + A_{-1}^N Bu(+0) + \sum_{k=0}^{N-1} A_{-1}^k (Bu(N-k+0) - Bu(N-k-0)).$$

As $\dot{\varphi}(-0) \in \mathbb{R}^n$ may be chosen arbitrarily, we obtain

$$\text{Im } A_{-1}^N \subset \text{Im} (B \ A_{-1}B \ \cdots \ A_{-1}^{N-1}B).$$

This may be written as

$$\text{rank} \begin{pmatrix} B & A_{-1}B & \cdots & A_{-1}^{N-1}B & A_{-1}^N \end{pmatrix} = \text{rank} \begin{pmatrix} B & A_{-1}B & \cdots & A_{-1}^{N-1}B \end{pmatrix}$$

By the Hamilton-Cayley theorem, this gives

$$\text{rank} \begin{pmatrix} B & A_{-1}B & \cdots & A_{-1}^{n-1}B & A_{-1}^n \end{pmatrix} = \text{rank} \begin{pmatrix} B & A_{-1}B & \cdots & A_{-1}^{n-1}B \end{pmatrix}.$$

Now, using Lemma 2.10, we obtain condition 2.2.

STEP 2. Let us assume that statement 2 of the theorem is verified. We want to show that the system is completely stabilizable by a feedback law of the form (11).

Let us fix an arbitrary $\omega \in \mathbb{R}$. As all the non-zero poles of the matrix A_{-1} are controllable by condition 2.2, then a matrix F_{-1} can be found such that the spectrum $\sigma(A_{-1} + BF_{-1})$ verifies

$$\forall \mu \in \sigma(A_{-1} + BF_{-1}), \mu \neq 0, \quad \ln |\mu| < \omega.$$

Consider now the neutral type system

$$\dot{z}(t) = (A_{-1} + BF_{-1})z_t + Lz_t + Bu \quad (14)$$

Let \mathcal{A}_1 be the generator of system (14). From the structure of the spectrum of neutral type systems like (1), we have only a finite number of eigenvalues $\lambda \in \sigma(\mathcal{A}_1)$ such that $\text{Re } \lambda \geq \omega$. Now, using condition 2.1 of the theorem and the fact that

$$\text{rank} (\mu I - A_{-1} \quad B) = n,$$

for all μ such that $\ln |\mu| \geq \omega$ (this means that $\sigma(A_{-1})$ is strictly included in a circle of radius e^ω), a feedback

$$u(t) = F_1 z_t(\cdot) = \int_{-1}^0 [F_2(\theta) \dot{z}(t+\theta) + F_3(\theta) z(t+\theta)] d\theta$$

can be found (see for example [15, 17, 22, 25]) such that all the eigenvalues λ of the system

$$\dot{z}(t) = (A_{-1} + BF_{-1})z_t + (L + BF_1)z_t$$

verify $\text{Re } \lambda < \omega$. If we denote by \mathcal{F} the global feedback

$$u(t) = F_{-1} \dot{z}(t-1) + F_1 z_t(\cdot),$$

then we obtain

$$\left\| e^{(\mathcal{A} + \mathcal{BF})t} \right\| \leq M e^{\omega t}, \quad M \geq 1.$$

Since ω has been arbitrarily taken, this means that the system is completely stabilizable by a feedback of type (11).

STEP 3. Suppose now that the system is completely stabilizable, let us show that the system is exactly null controllable. For technical reasons we need to show that condition 2.2 is verified.

Suppose that for some $\mu_0 \neq 0$ condition 2.1 is not verified. This means that for all possible $m \times n$ matrix F_{-1} , we have $\mu_0 \in \sigma(A_{-1} + BF_{-1})$. Then, for any feedback law \mathcal{F} of the type (11), we have

$$\text{Re } \sigma(\mathcal{A} + \mathcal{BF}) \geq \ln |\mu_0|.$$

This means that the system is not completely stabilizable, which contradicts our hypothesis.

Condition 2.2 implies (see for example [33]) that there is a matrix F_{-1} such that

$$\sigma(A_{-1} + BF_{-1}) = \{0\},$$

because all the non-zero eigenvalues are controllable. Let \mathcal{A}_1 be the semigroup corresponding to the neutral type system

$$\dot{z}(t) = (A_{-1} + BF_{-1})z(t-1) + Lz_t(\cdot) + Bu(t). \quad (15)$$

It is not difficult to show that condition 2.1 is verified for this system (15):

$$\text{rank} \begin{pmatrix} \Delta_{\mathcal{A}_1}(\lambda) & B \end{pmatrix} = n, \quad \forall \lambda \in \mathbb{C}.$$

Condition 2.2 is trivially verified because $\sigma(A_{-1} + BF_{-1})$ is reduced to the singleton $\{0\}$.

It may occur at this step, that the obtained state operator \mathcal{A}_1 has an empty spectrum. This implies that the exponential growth of the corresponding semigroup $e^{\mathcal{A}_1 t}$ is $\sup\{\text{Re } \lambda : \lambda \in \sigma(\mathcal{A}_1)\} = -\infty$. This means that for all $\omega > 0$, $e^{\omega t} e^{\mathcal{A}_1 t} \rightarrow 0$, as $t \rightarrow \infty$. This means that all solutions $z(t)$ of the corresponding neutral type system are *small* solutions and then [6] we have $z(t) = 0$ for $t \geq T_0 > n$. The original system is exactly null controllable by the feedback $u = F_{-1}\dot{z}$.

If not, in each half plane $\{\text{Re } \lambda \geq \omega\}$, there is a non zero finite number of eigenvalues. Hence, the system (15) is completely stabilizable by an \mathcal{A}_1 -bounded feedback of the form

$$Fz_t(\cdot) = \int_{-1}^0 [F_2(\theta)\dot{z}(t+\theta) + F_3(\theta)z(t+\theta)] d\theta. \quad (16)$$

This may be shown using different techniques: for example by exponential stabilizability (with an arbitrary decay rate) as in [15, 9], or using the solution of a linear-quadratic problem by means of a Riccati equation as in [17], where the neutral term is assumed to be exponentially stable, (see also [30], or by partial pole assignment as in [22, 25] for the finite part of the spectrum in the right half plane.

The feedback (16) is admissible and then by Theorem 2.9, system (15) is exactly null controllable. The fact that the spectrum is not empty, and the spectrum determined growth condition, implies that Assumption A is verified.

From the exact null controllability of the system (2.9) we deduce the exact null controllability of the original system (1) as has been proved for exact controllability from 0 in [24, 19, 20]). ■

Condition 2.2 has been conjectured for exact null controllability in [30, page 157] for a class of neutral type with one discrete delay.

If in Theorem 3.4 we put $A_{-1} = 0$, then condition 2.2 is automatically verified and we obtain the condition of exact null controllability obtained in [11] for a retarded system with discrete delays.

4 Final exact observability

The dual notion of exact null controllability in Hilbert space is the notion of final continuous observability. Sometimes the term continuous is replaced (by analogy) by the term exact. In [18], the duality between exact controllability and exact observability was analyzed. In the present section we give the result for null exact controllability and the corresponding notion of observability.

We consider the finite dimensional observation

$$y(t) = \mathcal{C}x(t) \quad (17)$$

where \mathcal{C} is a linear operator and $y(t) \in \mathbb{R}^p$ is a finite dimensional output. There are several ways to design the output operator \mathcal{C} [28, 30, 12]. One of our goals in this paper is to investigate how to design a minimal output operator like

$$\mathcal{C}x(t) = Cz(t) \quad \text{or} \quad \mathcal{C}x(t) = Cz(t-1), \quad (18)$$

where C is a $p \times n$ matrix. More general outputs, for example with several and/or distributed delays are not considered here. We want to use some results on exact controllability in order to analyze, by duality, the exact observability property in the infinite dimensional setting like, for example, in [32].

The operator \mathcal{C} defined in (18) is linear but not bounded in M_2 . However, in both cases it is admissible in the following sense:

$$\int_0^T \|\mathcal{C}e^{At}x_0\|_{\mathbb{R}^n}^2 dt \leq \kappa^2 \|x_0\|_{M_2}^2, \quad \forall x_0 \in D(\mathcal{A}),$$

$e^{At}x_0 \in D(\mathcal{A})$, $t \geq 0$ (see for example [16]). In fact, \mathcal{C} is admissible in the resolvent norm: $\|x_0\|_{-1} = \|(\lambda I - \mathcal{A})^{-1}x_0\|_{M_2}$, $\lambda \in \rho(\mathcal{A})$. This is due to the fact that \mathcal{C} is a closed operator and takes value in a finite dimensional space (see [32, Def. 4.3.1] and comments on this definition).

Definition 4.1 *Let \mathcal{K} be the output operator*

$$\mathcal{K} : M_2 \longrightarrow L_2(0, T; \mathbb{R}^p), \quad x_0 \longmapsto \mathcal{K}x_0 = \mathcal{C}e^{At}x_0.$$

System (1) is said to be exactly observable (or continuously observable [30]) if

$$\|\mathcal{K}x_0\|_{L_2} = \left(\int_0^T \|\mathcal{C}e^{At}x_0\|_{\mathbb{R}^p}^2 dt \right)^{\frac{1}{2}} \geq \gamma \|e^{AT}x_0\|_{M_2}, \quad (19)$$

for some constant $\gamma > 0$, and for all $x_0 \in D(\mathcal{A})$.

The exact (final) observability depends essentially on the topology of the space. We can expect that, the given neutral type system is not exactly observable if we consider $x_0 \in D(\mathcal{A})$, with the norm of the graph and no longer in the topology of M_2 . Taking into account the result on exact null controllability, it seems that (19) must be changed by taking a weaker norm for x_0 , namely the resolvent norm $\|(\lambda I - \mathcal{A})^{-1}x_0\|$ and considering the extension of the operator \mathcal{K} to the completion of the space with this norm. In fact, we obtain the final observability in the initial norm but we need some delay in the observation in the general case.

In order to use the duality between observability and controllability, we need the expression of the adjoint operator \mathcal{K}^* in the duality with respect to the pivot space M_2 in the embedding

$$X_1 \subset X = M_2 \subset X_{-1}, \quad (20)$$

where $X_1 = D(\mathcal{A})$ with the graph norm noted $\|x\|_1$ and X_{-1} the completion of the space M_2 with respect to the resolvent norm $\|x\|_{-1} = \|(\lambda I - \mathcal{A})^{-1}x\|_{M_2}$. The duality relation is

$$\langle \mathcal{K}x_0, u(\cdot) \rangle_{L_2(0, T; \mathbb{R}^p)} = \langle x_0, \mathcal{K}^*u(\cdot) \rangle_{X_1, X_{-1}^d}, \quad (21)$$

where X_{-1}^d is constructed as X_{-1} with \mathcal{A}^* instead of \mathcal{A} (see [32] for example).

Exact null controllability is dual with exact final observability in the corresponding spaces with the corresponding topologies. It is expected that the operator \mathcal{K}^* corresponds to a control operator for some adjoint system. However, The situation is not so simple, as it was pointed out in the paper [18], from which we take our main considerations on duality.

Proposition 4.2 ([22, 18]) *The adjoint operator \mathcal{A}^* is given by*

$$\mathcal{A}^* \begin{pmatrix} w \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} (A_2^*(0)w + \psi(0)) \\ -\frac{d[\psi(\theta) + A_2^*(\theta)w]}{d\theta} + A_3^*(\theta)w \end{pmatrix}, \quad (22)$$

with the domain $D(\mathcal{A}^*)$:

$$\{(w, \psi(\cdot)) : \psi(\theta) + A_2^*(\theta)w \in H^1, A_{-1}^* (A_2^*(0)w + \psi(0)) = \psi(-1) + A_2^*(-1)w\}.$$

Let x be a solution of the abstract equation

$$\dot{x} = \mathcal{A}^*x, \quad x(t) = \begin{pmatrix} w(t) \\ \psi_t(\theta) \end{pmatrix}. \quad (23)$$

Then the function $w(t)$ is the solution of the neutral type equation

$$\dot{w}(t+1) = A_{-1}^* \dot{w}(t) + \int_{-1}^0 [A_2^*(\tau) \dot{w}(t+1+\tau) + A_3^*(\tau)w(t+1+\tau)] d\tau. \quad (24)$$

This means that the form of the adjoint system is not a simple transposition of the initial one (1). Let us now specify the relation between the solutions of the neutral type equation (24) related to the adjoint system (23) and the transposed neutral type equation

$$\dot{z}(t) = A_{-1}^* \dot{z}(t-1) + \int_{-1}^0 [A_2^*(\tau) \dot{z}(t+\tau) + A_3^*(\tau)z(t+\tau)] d\tau + C^*u(t), \quad (25)$$

with initial $z_0(\theta)$. Let \mathcal{A}^\dagger be the infinitesimal generator of the semigroup corresponding to equation (25).

Let us put

$$\begin{pmatrix} w(t) \\ \psi_t(\theta) \end{pmatrix} = e^{\mathcal{A}^*t} \xi_0 = e^{\mathcal{A}^*t} \begin{pmatrix} w(0) \\ \psi_0(\theta) \end{pmatrix},$$

and

$$\begin{pmatrix} v(t) \\ z_t(\theta) \end{pmatrix} = \begin{pmatrix} w(t+1) - A_{-1}^* w(t) \\ w(t+1+\theta) \end{pmatrix} = e^{\mathcal{A}^\dagger t} \begin{pmatrix} v(0) \\ z_0(\theta) \end{pmatrix} = e^{\mathcal{A}^\dagger t} x_0,$$

where $z_0(\theta) = w(\theta+1)$ and $v(0) = z_0(0) - A_{-1}z_0(-1)$. We can give the explicit relation between the initial conditions ξ_0 and x_0 :

$$\xi_0 = \begin{pmatrix} w(0) \\ \psi_0(\theta) \end{pmatrix}, \quad x_0 = \begin{pmatrix} v(0) \\ z_0(\theta) \end{pmatrix}.$$

The formal relation between these vectors is

$$\xi_0 = \begin{pmatrix} w(0) \\ \psi_0(\theta) \end{pmatrix} = \Phi x_0 = \Phi \begin{pmatrix} w(1) - A_{-1}w(0) \\ w(\theta+1) \end{pmatrix}.$$

and we have the following result.

Theorem 4.3 [18] *The operator Φ representing the relation between initial conditions x_0 and ξ_0 corresponding to neutral type systems (23) – (24) and (25) is linear bounded and bounded invertible from X_1^d to M_2 , where X_1^d is $D(\mathcal{A}^*)$ with the graph norm.*

Let us now consider the reachability operator \mathcal{R}^\dagger of system (25):

$$\mathcal{R}^\dagger u(\cdot) = \int_0^T e^{\mathcal{A}^\dagger(T-\tau)} \mathcal{C}^\dagger u(\tau) d\tau, \quad \mathcal{C}^\dagger = \begin{pmatrix} C \\ 0 \end{pmatrix}.$$

Then the operator \mathcal{K} may be written using \mathcal{R}^\dagger and the semigroup $e^{\mathcal{A}^\dagger}$ of system (25) as follows (see [18]):

$$\mathcal{K}x_0 = \begin{cases} \mathcal{R}_T^{\dagger*} \Phi x_0 & \text{if } \mathcal{C}x(t) = Cz(t-1), \\ \mathcal{R}_T^{\dagger*} e^{\mathcal{A}^\dagger*} \Phi x_0 & \text{if } \mathcal{C}x(t) = Cz(t). \end{cases} \quad (26)$$

We can now formulate the main result of this section.

Theorem 4.4 *System (1) with the output $y = Cz(t-1)$ is exactly (continuously) finally observable if and only if system (25) is exactly null controllable. The characterization of this property is given by two conditions:*

1. $\text{Ker} \begin{pmatrix} \Delta^{\mathcal{A}(\lambda)} \\ C \end{pmatrix} = \{0\}$ for all $\lambda \in \mathbb{C}$,
2. $\text{Ker} \begin{pmatrix} \lambda I - A_{-1} \\ C \end{pmatrix} = \{0\}$ for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$.

Proof. According to relation (26) we have

$$\|\mathcal{K}x_0\|_{L_2} = \left(\int_0^T \left\| C^* e^{\mathcal{A}^{\dagger*}(T-\tau)} \Phi x_0 \right\|^2 d\tau \right)^{\frac{1}{2}}.$$

As system (25) is exactly null controllable, we obtain

$$\|\mathcal{K}x_0\|_{L_2} \geq \delta \left\| e^{\mathcal{A}^{\dagger*}T} \Phi x_0 \right\|,$$

for all $x_0 \in D(\mathcal{A})$. It is easy to see from [18] that

$$e^{\mathcal{A}^{\dagger*}T} \Phi x_0 = \Phi e^{\mathcal{A}^{\dagger*}(T-\tau)} x_0 = e^{\mathcal{A}^T} x_0.$$

This gives

$$\|\mathcal{K}x_0\|_{L_2} \geq \delta \|e^{\mathcal{A}^T} x_0\|,$$

which means that continuous final observability holds. ■

For the case of the output $y = Cz(t)$ we cannot say anything if $\det(\mathcal{A}_{-1}) = 0$. If \mathcal{A}_{-1} is not singular, then $e^{\mathcal{A}t}$ is a group and exact final observability coincides with exact observability [18].

5 Examples

All examples are given in the form of a system with one discrete delay:

$$\dot{z}(t) = A_{-1}z(t-1) + A_0z(t) + A_1z(t-1) + Bu(t).$$

Example 5.1

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = 0, \quad A_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We have

$$\forall \lambda \in \mathbb{C}, \quad \text{rank} \begin{pmatrix} \Delta_{\mathcal{A}}(\lambda) & B \end{pmatrix} = \begin{pmatrix} \lambda - \lambda e^{-\lambda} & 0 & 1 \\ -1 & \lambda & 0 \end{pmatrix} = 2,$$

and

$$\forall \lambda \in \mathbb{C}, \quad \lambda \neq 0, \quad \text{rank} (\lambda I - A_{-1} \quad B) = \begin{pmatrix} \lambda - 1 & 0 & 1 \\ 0 & \lambda & 0 \end{pmatrix} = 2.$$

The system is exactly null controllable. Consider now the transposed system

$$\begin{cases} \dot{z}_1(t) &= z_2(t) \\ \dot{z}_2(t) &= z_2(t-1) \end{cases}$$

This system is continuously finally observable by the feedback $y = z_1(t-1)$ by Theorem 4.4.

Example 5.2

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = 0, \quad A_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We can see that the spectrum of A_{-1} is reduced to $\{0\}$. Condition 2.2 of Theorem 3.4 is automatically verified. On the other hand, we have:

$$\forall \lambda \in \mathbb{C}, \quad \text{rank} \begin{pmatrix} \Delta_{\mathcal{A}}(\lambda) & B \end{pmatrix} = \begin{pmatrix} \lambda & -\lambda e^{-\lambda} - 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix} = 2.$$

The system is exactly controllable because the par (A_{-1}, B) is controllable. The transposed system

$$\begin{cases} \dot{z}_1(t) &= 0 \\ \dot{z}_2(t) &= z_1(t-1) + z_1(t) \end{cases}$$

is continuously observable with the output $y = z_2(t-1)$ but not with $y(t) = z_2(t)$.

Example 5.3

This example was given in [12] for exact null controllability and continuous final observability.

$$A_0 = 0, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

In fact this system is exactly observable by the output $y = Cz(t-1)$ and the transposed system is exactly controllable because

$$\forall \lambda \in \mathbb{C}, \quad \text{rank} (\lambda I - A_{-1}^* \quad B^*) = \text{rank} \begin{pmatrix} \lambda & 0 & 1 \\ 1 & \lambda - 1 & 0 \end{pmatrix} = 2,$$

However, the initial system is not observable by the output $y = Cz(t) = z_1(t)$, because the initial function $z_0(\theta), \theta \in [0, 1[$ cannot be determined.

6 Conclusion

Using a general result on the equivalence between exact null controllability and complete stabilizability of abstract systems in Hilbert spaces, a characterization of these properties has been given for a large class of linear neutral type systems. This also enables the final continuous observability of such systems to be characterized. The following step is to extend such results to the problem of detectability, dual with stabilizability.

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